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The Integer L-Shaped Method for Stochastic Integer Programs

by

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Abstract

In this paper, a general branch-and-cut procedure for stochastic integer programs with first stage binary variables is presented. It is shown to provide a finite exact algorithm for a number of stochastic integer programs, even in the presence of binary variables or continuous random variables in the second stage.

Key words : Stochastic integer programming, L-shaped method, branch-and-cut.

Résumé

On décrit dans cet article une procédure générale de "branch-and-cut" pour la résolution de programmes stochastiques en nombres entiers, avec variables binaires en première période. La méthode proposée constitue un algorithme exact fini, applicable à une classe générale de programmes stochastiques en nombres entiers, même lorsque ceux-ci comportent des variables binaires ou des variables aléatoires continues en seconde période.

Mots-clés : Programmation stochastique en nombres entiers, méthode "L-shaped", "branch-and-cut".

1. Introduction

Stochastic integer programs have a well-deserved reputation for being computationally intractable. In fact, they combine two types of programs which by themselves are often difficult to solve. Stochastic programs are modeled in two stages. In the first stage, a decision has to be taken without full information on some random variable. When the value taken by the random variable is known, corrective or recourse actions can be taken in the second stage. Stochastic integer programs are those for which some variables are restricted to be integer, either in the first stage or in the second stage.

To the best of our knowledge, the first proposed solution method was a cutting planes algorithm by Wollmer (1980) for the case where second stage decision variables are continuous and the random variable is discrete. In a similar vein, Louveaux (1986) has shown that multistage programs having the block-separability property are equivalent to two-stage problems of the type solved by Wollmer. Other authors have studied properties of the expected value of the second stage program when it involves integer decision variables. General results concerning the continuity of this function have been obtained by Stougie (1985). This type of research has also been done for specific applications: e.g. Jaillet (1988), or Laporte and Louveaux (1990) for the analysis of some vehicle routing problems. Several studies have addressed the behaviour of various heuristics for

specific problems (see, e.g., chapters 8 and 21 in Ermoliev and Wets (1988), also containing general results on stochastic programming). One line of research is the study of the asymptotic optimality of some heuristics, based on the probabilistic analysis of algorithms. However, Psaraftis (1984) has shown through an example that optimality is truly asymptotic in the sense that it only holds for problems of extremely large scale. The need for a practical solution method of general applicability still exists.

The object of this paper is to provide such an algorithm for problems with first stage binary decision variables. The random vectors and the second stage decision variables may be discrete or continuous, and the application of the method varies accordingly.

The structure of the paper is as follows. In Section 2, we review the definition of two-stage stochastic linear programs with recourse. We also provide a simple taxonomy for the various types of stochastic integer programs. The general branch-and-cut procedure, called "Integer L-shaped" due to its similarity with the well-known method of Van Slyke and Wets (1969) for continuous programs, is presented in Section 3. We prove its finite convergence in the case of first stage binary decision variables. In Section 4, we discuss the implementation of the proposed method to problems with various types of random vectors and second stage decision variables. The conclusion follows in Section 5.

2. Two-Stage Stochastic Linear Programs with Recourse

The general formulation of a two-stage stochastic linear program with fixed recourse is to find

$$\min z = cx + E_\xi \min \{q(\omega)y \mid Wy = h(\omega) - T(\omega)x, y \in Y\} \qquad (1)$$

$$s.t. \quad Ax = b$$
$$x \in X,$$

where c is a known vector in $\Re^{n_1}$, b a known vector in $\Re^{m_1}$, ξ a random N-vector defined on the probability space $(\Omega, \mathcal{A}, P)$, and A and W are known matrices of size $m_1 \times n_1$ and $m_2 \times n_2$, respectively. For each $\omega, T(\omega), q(\omega)$ and $h(\omega)$ have conformable sizes. Piecing together the stochastic components of the problem, we obtain a vector $\xi(\omega) = (q(\omega)h(\omega), T_1(\omega), ..., T_{m_2}(\omega))$ with $N = n_2 + m_2 + (m_2 \times n_1)$ components, where $T_i(\omega)$ is the i^{th} row of $T(\omega)$. Transposes have been eliminated for simplicity. E_ξ represents the mathematical expectation with respect to ξ. Let also $\Xi \subseteq \Re^N$ be the support of ξ, i.e. the smallest closed subset in $\Re^N$ such that $P(\xi \in \Xi) = 1$. If ξ has finite support, there exists a discrete probability distribution which describes ξ.

The above problem is equivalent to the so-called deterministic equivalent program:

$$\min \ z = cx + Q(x) \qquad (2)$$

$$s.t. \qquad Ax = b, x \in X$$

where $$Q(x) = E_\xi Q(x, \xi(\omega)) \tag{3}$$

and $$Q(x, \xi(\omega)) = \min_y \{q(\omega)y \mid Wy = h(\omega) - T(\omega)x, y \in Y\}. \tag{4}$$

The function $Q(x)$ is called the second stage value function. Of particular interest in this paper are the cases when the sets X or Y define restrictions requiring some variables to be integer.

In order to characterize the various possible situations, we use the notation $a \mid b \mid c$ to represent a particular class of two-stage programs. The first letter characterizes the first stage variables, and is either C if all first stage variables are continuous, B if they are all binary, or M when some first stage variables are continuous and others are binary (mixed integer problems). Similarly, the second letter characterizes the second stage variables with an identical representation. The third letter is either D when ξ has a finite support (discrete distribution), or C when ξ is a continuous random variable. Thus, $C \mid C \mid D$ represents a two-stage continuous stochastic program with a discrete distribution. For this case, well-known characterization theorems and solution algorithms exist (see, e.g., Wets, 1983). For some classes of problems, it is also useful to know whether or not they possess the block-separable property (see Louveaux, 1986) and this is denoted by an additional $\mid s$. For example, $M \mid M \mid D \mid s$ is a stochastic program with mixed integer first and second stage, no distinction is made between integer and binary variables, since integer variables can be transformed into binary variables.

3. The Integer L-Shaped Method

Let $\overline{X}$ be the polytope defined by the set of constraints in X, other than those defining the first stage variables type. Hence $X = \overline{X} \cap \Re_+^{n_1}$ for $C \mid * \mid *$ problems, $X = \overline{X} \cap Z_+^{n_1}$ for $B \mid * \mid *$ problems, and similarly for $M \mid * \mid *$ problems. In this section, we assume that at least some first stage variables are binary. For simplicity, all variables are non-negative.

At a given stage of the method, we consider the so-called *current problem*:

$$\text{(CP)} \qquad \min \quad cx + \theta \tag{5}$$

$$s.t. \quad Ax = b \tag{6}$$

$$D_k x \geq d_k \qquad (k = 1, ..., s) \tag{7}$$

$$E_k x + \theta \geq e_k \qquad (k = 1, ..., t) \tag{8}$$

$$x \geq 0, \theta \in \Re. \tag{9}$$

In this current problem, constraints (7) are used to give an appropriate representation of $\overline{X}$. Usually, $\overline{X}$ is relaxed at the initial stage, and additional constraints (7) are added when needed. These constraints are referred to as *feasibility cuts*. A set of feasibility cuts is said to be valid if there exists some finite value s such that $x \in \overline{X}$ if and only if $\{D_k x \geq d_k, k = 1, ..., s\}$. Examples of such relaxed constraints are induced constraints for second stage feasibility, or subtour elimination constraints in vehicle routing problems.

Variable θ, along with the constraints (8), are used to express θ as an appropriate approximation of $Q(x)$. Constraints (8) are referred to as *optimality cuts*. A set of t optimality cuts is said to be valid if for all $x \in X$, $(x, \theta) \in \{(x, \theta) \mid E_k x + \theta \geq e_k, k = 1, ..., t\}$ implies $\theta \geq Q(x)$. In practice, it is not required to have $\theta \geq Q(x)$ for all feasible (x, θ), but simply for the optimal (x^*, θ^*).

We first present the Integer L-shaped method in general terms.

Step 0 : $s := t := \nu := 0, \overline{z} := \infty$; the only pendant node corresponds to the initial current problem.

Step 1 : Select some pendant node in the list; if none exists, stop.

Step 2 : Set $\nu := \nu + 1$; solve the current problem. If the current problem has no feasible solution, fathom the current node; go to Step 1. Otherwise, let (x^ν, θ^ν) be an optimal solution. (If $t = 0, \theta$ is set equal to $-\infty$ or to an appropriate lower bound and is ignored in the computation).

Step 3 : Check for any relaxed constraint violation. If one exists, add one feasibility cut (7), set $s := s + 1$, and return to Step 2. If $cx^\nu + \theta^\nu > \overline{z}$, fathom the current problem and return to Step 1.

Step 4 : Check for integrality restrictions. If one is violated, create two new branches following the usual branch and cut procedure; append the new nodes to the list of pendant nodes; return to Step 1.

Step 5 : Compute $Q(x^\nu)$ and $z^\nu = cx^\nu + Q(x^\nu)$. If $z^\nu < \overline{z}$, then update $\overline{z}$.

Step 6 : If $\theta^\nu \geq Q(x^\nu)$, then fathom the current node and return to Step 1. Otherwise impose optimality cuts (8) and return to Step 2. ■

As can be seen, the fathoming rules are different from usual. In particular, nodes are not necessarily fathomed when integrality conditions are satisfied. The following proposition follows immediately from the fathoming rules and from the definitions of valid sets of cuts.

Proposition 1. For any problem for which a valid set of feasibility cuts and a valid set of optimality cuts exist, the Integer L-shaped method yields an optimal solution (when one exists) in a finite number of steps. ■

For most combinatorial problems of interest, valid feasibility cuts can be borrowed directly from the deterministic setting. As for optimality cuts, a valid set is known to exist for two-stage stochastic linear programs in continuous decision variables with discrete random variables. In the L-shaped method (Van Slyke and Wets, 1969), these are defined using the optimal simplex multipliers of the second stage recourse problem. It follows that the Integer L-shaped method is applicable to all problems belonging to the $* \mid C \mid D$ class. It offers an interesting alternative to the cutting planes method used by Wollmer (1980). Moreover, as a consequence of a Proposition by Louveaux (1986, Section 5), the method is valid for $* \mid M \mid D \mid s$ problems. Note that this property also applies to multistage programs. For several families of deterministic integer problems, branch-and-

bound techniques are known to perform better than cutting planes and we can therefore expect the same to hold for stochastic integer problems. However, our purpose here is not to make further developments in that direction, but rather to derive optimality cuts for problems with first stage binary decision variables. Two assumptions are required.

Assumption 1. Given a binary first stage vector of decision variables x, the function $Q(x)$ is computable.

Assumption 2. There exists a finite value L satisfying $L \leq \min_{x}\{Q(x) \mid Ax = b, x \in \{0,1\}^{n_1}\}$.

Since the first stage decision variables are binary, there are only a finite number of feasible first stage solutions. Let $r = 1, ..., R$ index these feasible solutions.

Proposition 2. Let $x_i = 1, i \in S$ and $x_i = 0, i \notin S$ be the r^{th} feasible solution, and θ_r the corresponding expected second-stage value. Define the optimality cut as

$$\theta \geq (\theta_r - L)\left(\sum_{i \in S} x_i - \sum_{i \notin S} x_i\right) - (\theta_r - L)(|S| - 1) + L, \qquad (10)$$

where $|S|$ is the cardinality of S. Then, the set of optimality cuts (10) defined for all first stage feasible solutions is a valid set of optimality cuts.

Proof. The quantity $\sum_{i \in S} x_i - \sum_{i \notin S} x_i$ is always less than or equal to $|S|$. It takes the value $|S|$ only when x is the r^{th} feasible solution. Now, when $\sum_{i \in S} x_i - \sum_{i \notin S} x_i$ is equal to $|S|$, the right-hand side takes the value θ_r, and otherwise the right-hand side takes a value less than or equal to L. A set of valid cuts is therefore obtained by imposing one such constraint for each first-stage feasible solution. ■

4. Implementation

The applicability of the Integer L-shaped method is directly related to Assumptions 1 and 2 being satisfied. Also, as for all branch-and-cut methods, its efficiency is greatly improved if in addition to optimality cuts, valid constraints can be generated to provide a lower bounding functional on $Q(x)$. This is now discussed.

The validity of Assumption 1 is obviously problem dependent. It generally holds when the random vector has a discrete support of small cardinality, especially if only a limited number of scenarios are considered in the second stage. When the random vector is continuous, one can resort to approximation methods normally used in continuous stochastic optimization (Birge and Wets, 1986), such as discretizing continuous random variables. Moreover, there exists a wide variety of problems for which $Q(x)$ can be easily computed. Several interesting examples are found in the literature on vehicle routing problems (VRPs). These include the VRP with stochastic demands (Dror and Trudeau, 1986), the proba-

bilistic traveling salesman problem (Jaillet, 1988), and the VRP with stochastic travel times (Laporte, Louveaux and Mercure, 1991).

We now discuss Assumption 2 and the derivation of a lower bounding functional on $Q(x)$. Let

$$Q(x,\xi) = \inf_y \{qy \mid Wy = h - Tx, y \geq 0 \text{ and integer}\}. \tag{11}$$

As usual, $\xi = (h,T,q)$, $Q(x,\xi)$ is set equal to infinity if no second stage solution exists, and to minus infinity if the problem is unbounded. Also define

$$R(x,\xi) = \inf_y \{qy \mid Wy = h - Tx, y \geq 0\}, \tag{12}$$

the solution to the continuous relaxation of $Q(x,\xi)$, with similar conventions for the infeasible and unbounded cases. The expectations of these two functions are

$$Q(x) = E_\xi Q(x,\xi) \tag{13}$$

and

$$R(x) = E_\xi R(x,\xi), \tag{14}$$

respectively, with the classical convention "$(+\infty) + (-\infty) = (+\infty)$".

Proposition 3. Any continuous L-shaped optimality cut is a valid lower bound on $Q(x)$.

Proof. First note that $Q(x,\xi) \geq R(x,\xi)$ holds for all x and ξ. Indeed, any feasible solution for (11) is feasible for (12), so that $R(x,\xi) = \infty$ only if $Q(x,\xi) = \infty$. If $Q(x,\xi)$ is finite, so is $R(x,\xi)$ and the result holds. Taking expectations on both sides yields $Q(x) \geq R(x)$. The conclusion follows by applying the continuous L-shaped optimality cut to $R(x)$, namely

$$Q(x) \geq R\left(x^k\right) + \partial R\left(x^k\right)\left(x - x^k\right), \tag{15}$$

where x^k is any feasible first stage solution, and $\partial R(x^k)$ is a subgradient of R at x^k. ■

Corollary 4. A valid cut for $Q(x)$ is given by

$$\theta \geq R(x^k) + \partial R(x^k)(x - x^k). \tag{16}$$

■

Corollary 5. A valid lower bound on $Q(x)$ is given by

$$L = \min_{x,\theta} \{\theta \mid Ax = b, 0 \leq x \leq 1 \text{ and } (\theta, x) \text{ satisfies (16) for } k = 1, ..., s\}, \tag{17}$$

for any finite value $s \geq 1$. ■

Note that (17) defines a finite lower bound on $Q(x)$ for any $s \geq 1$, as long as a feasible x^k exists. Indeed, in such a case, $R(x^k)$ is finite and the last term of (16) is bounded below since $0 \leq x \leq 1$. Another observation is that (16) plays a different role from (10). An optimality cut ensures that $\theta \geq Q(x^k)$, but only at the particular point x^k at which it has been defined. For any other integer point, it reduces to the weaker constraint $\theta \geq L$. On the other hand, the cut defined by (16) is more general in that it may provide a non-trivial lower bound on $Q(x)$,

for several values x. However, when applied to the point x^k, at which it has been defined, it only guarantees that $\theta \geq R(x^k)$. In general, $R(x^k) < Q(x^k)$ since these two values coincide only if the solution to (11) and (12) at x^k reveals no integrality gap for all values of ξ. As such, cuts (16) do not provide a valid finite set of optimality cuts.

Finally, as for the computation of $Q(x)$, there also exist several problems for which the lower bound L and a lower bounding functional can easily be determined by arguments other than those used to obtain Corollaries 4 and 5. Applications of these principles and implementation details can be found in two recent studies on the VRP with stochastic travel times (Laporte, Louveaux and Mercure, 1991) and on a location problem with stochastic demands (Laporte, Louveaux and Van hamme, 1990).

5. Conclusion

We have described in this paper a general solution procedure for a class of stochastic programming problems with binary first stage decision variables. The method is of wide applicability, but its performance is improved if it is tailored to the shape of the problem at hand.

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